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ON THE LOWER BOUND FOR THE INJECTIVITY RADIUS OF 1/4-PINCHED RIEMANNIAN MANIFOLDS

JEFF CHEEGER & DETLEF GROMOLL

The purpose of this paper is to prove that if M is a compact simply connected riemannian manifold whose sectional curvature K satisfies $1/4 \leqslant K \leqslant 1$, then every closed geodesic has length $\geqslant 2\pi$, or equivalently, the injectivity radius of the exponential map is $\geqslant \pi$.

This result, which for example, is necessary for the proof of Berger's rigidity theorem [1], [2], is well known in the even dimensional case (assuming only $0 < K \leqslant 1$), and in the odd dimensional case [6] provided one assumes $1/4 < K \leqslant 1$. The case $1/4 \leqslant K \leqslant 1$ is treated in [7], but the argument seems to be somewhat unclear at several points. Our proof is in spirit a modification of the ideas in [7].

We will use some fairly general standard facts about lifting curves by the exponential map $\exp: TM \to M$. Our basic references are [2] and [4]. For $m \in M$, $\exp_m$ is the restriction of exp to the tangent space M_m of M at m. All curves under consideration will be continuous and parametrized on [0, 1]. Let P denote the set of all curves in M, and $\tilde{P}$ the set of vertical curves $\tilde{c}$ in TM emanating from the zero section, i.e., $c = \pi \circ \tilde{c}$ is constant, $\tilde{c}(0) = 0 \in M_{c(0)}$, where $\pi: TM \to M$ is the natural projection. Throughout this paper, we will use the topology of uniform convergence for P, $\tilde{P}$, and any subset.

Let $\operatorname{Exp}: \tilde{P} \to P$ denote the continuous map induced by exp, $\operatorname{Exp}(\tilde{c}) = \exp \circ \tilde{c}$. The image of Exp is the set of *liftable* curves; any $\tilde{c} \in \tilde{P}$ is a *lift* of $c = \operatorname{Exp}(\tilde{c})$. We call $\tilde{c}$ a *regular lift* if $\exp_m$ has maximal rank $n = \dim M$ at all points $\tilde{c}(t)$, $0 \leqslant t \leqslant 1$, $m = \pi \circ \tilde{c}(0)$, or equivalently, $\pi \times \exp$ has maximal rank $2n$ at all $\tilde{c}(t)$. Clearly, any $c \in P$ has a unique lift if it has a regular lift, but in general need not have a regular lift or any lift at all. All regular lifts form an open subset $\tilde{Q} \subset \tilde{P}$, and Exp imbeds $\tilde{Q}$ homeomorphically onto an *open* subset $Q \subset P$. We are particularly interested in the set of closed curves $P_0 \subset P$. Lifts of closed curves, if they exist, need not be closed. P_0 certainly

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contains the Exp-image of the closed curves $\tilde{P}_0$ in $\tilde{P}$. Finally, let $\tilde{Q}_0 = \tilde{P}_0 \cap \tilde{Q}$ denote the set of closed regular lifts, which is open in $\tilde{P}_0$. It follows from the above that Exp maps $\tilde{Q}_0$ homeomorphically onto an open subset Q_0 of P_0. We conclude this discussion with two simple

Remarks. (1) Any geodesic $c \in P$ always has a canonical (radial) lift $\tilde{c}$, regular or not, which is never closed.

(2) Suppose for some $m \in M$ and $r > 0$, $\exp_m$ is nonsingular on the open ball $U_r(m) = \{v \mid \|v\| < r\} \subset M_m$. Let $c \in P$ be piecewise differentiable, $c(0) = m$, of length $L(c) < r$. Then $c \in Q$, so c has a unique regular lift $\tilde{c} = \mathrm{Exp}^{-1}(c)$, and $\tilde{c}(t) \in U_r$, $0 \leqslant t \leqslant 1$. In essence, this is a consequence of the Gauss lemma.

Since we have to employ some arguments from Morse theory, we will also work with the standard approximations of P_0 by the finite dimensional subspaces $\Omega^{\leqslant a}$ of all closed broken geodesics c of energy $E(c) \leqslant a$, with break point parameters chosen as usual, sufficiently fine and fixed for an energy level $b \geqslant a$ large enough in a given situation. Let $\Omega^{<a}$ denote the open subset of all curves in $\Omega^{\leqslant a}$ whose energy is strictly less than a. Observe that $\Omega^{\leqslant a}$ need not be the closure of $\Omega^{<a}$. The critical points of the energy function E on these free loop spaces are precisely smoothly closed geodesics in M.

We now prove a crucial fact about limits of closed lifts which is quite general. For any curve c and $0 \leqslant s \leqslant 1$, let c_s be the curve given by $c_s(t) = c(st)$, and let c^- denote the reversed curve, $c^-(t) = c(1-t)$. By c_s^- we will mean $(c_s)^-$.

Lemma 1 (*Lifting lemma*). *Suppose, for some $r > 0$ and all $m \in M$, $\exp_m$ is nonsingular on the open ball $U_r(m) \subset M_m$. Let $c \in P_0$ be in the closure of $\mathrm{Exp}(\tilde{P}_0) \cap \Omega^{\leqslant 4r^2}$. Then either $c \in Q_0$ (and thus is not a closed geodesic if nonconstant), or both $c_{1/2}$ and $c_{1/2}^-$ are geodesics of length r with conjugate end points. Furthermore, $C = Q_0 \cap \Omega^{<4r^2}$ is a connected component of $\Omega^{<4r^2}$.*

Remarks. (3) $\Omega^{<4r^2}$ is a manifold, so its connected components are precisely its pathwise connected components. But $\Omega^{\leqslant 4r^2}$ need not be a manifold (with boundary) if $4r^2$ is a critical value of the energy E. To avoid difficulties, we will mostly work with connected, rather than pathwise connected components.

(4) If we replace the assumption $\exp_m | U_r(m)$ nonsingular for all m by the assumption $\exp_{c(0)} | U_r(c(0))$ is nonsingular and leave the rest of the hypothesis unchanged, then the conclusions of Lemma 1 continue to hold for the loop space at $c(0)$. But this is not sufficient for our purposes.

Proof. We can assume that at least one of the two branches $c_{1/2}$, $c_{1/2}^-$ (say $c_{1/2}$) is not a geodesic, or if so, is free of conjugate points. Note that

$L(c_{1/2}) \leqslant \sqrt{E(c_{1/2})} \leqslant r$, with equality iff $c_{1/2}$ is a geodesic of length r. Therefore, by Remarks 1 and 2, $c_{1/2} \in Q$. But Q is open, so in fact $c_s \in Q$ for some $1/2 < s \leqslant 1$. Then, again by Remark 2, $c_{1-s}^- \in Q$. By hypothesis, there exists a sequence $\tilde{c}_i \in \tilde{P}_0$ with $\operatorname{Exp}(\tilde{c}_i) = c_i$ converging to c. Now $(\operatorname{Exp}|\tilde{Q})^{-1}$ is well-defined and continuous, so it follows that $\operatorname{Exp}^{-1}(c_{i,s}) = \tilde{c}_{i,s} \to \tilde{c}_s$, and $\operatorname{Exp}^{-1}(c_{i,1-s}^-) = \tilde{c}_{i,1-s}^- \to \tilde{c}_{1-s}^-$.

Since $\tilde{c}_{i,s}(1) = \tilde{c}_{i,1-s}^-(1)$ for all i, we have $\tilde{c}_s(1) = \tilde{c}_{1-s}^-(1)$. Thus $\tilde{c}_i$ converges to some $\tilde{c} \in \tilde{Q}_0$, and $\operatorname{Exp}(\tilde{c}) = c \in Q_0$. $C = Q_0 \cap \Omega^{<4r^2}$ is open, and we have just shown it is also closed in $\Omega^{<4r^2}$. Therefore C is a union of connected components of $\Omega^{<4r^2}$. If C_1 is any component of C, then C_1 contains a closed geodesic on which E takes its minimum value. Since C contains no nontrival closed geodesics, it follows that $C = C_1$ is connected.

We need the following result from Morse theory.

Lemma 2. *Let f be a smooth function on the differentiable manifold X of dimension k, and p a possibly degenerate critical point of index $\geqslant 2$ (or a regular point) with $f(p) = a$. Then there exists a neighborhood N of p such that $N \cap X^{<a}$ is (pathwise) connected and dense in $N \cap X^{\leqslant a}$.*

Proof. We may assume $X = \mathbf{R}^k$, $p = 0$, $f(0) = a = 0$. If 0 if a regular point, our claim is trivial. Otherwise, according to the generalized Morse Lemma in [5], we have, after a change of coordinates near the origin in $\mathbf{R}^k = \mathbf{R}^2 \times \mathbf{R}^{k-2}$,

$$f(x, y) = -\|x\|^2 + g(y),$$

on some neighborhood U, where g is a smooth function in $\mathbf{R}^{k-2}$. Now choose $d, r > 0$ such that $(x, y) \in U$ and $g(y) < d^2$ on

$$N = \{(x, y) \mid \|x\|^2 \leqslant d^2, \|y\|^2 \leqslant r^2\}.$$

Let $q_0 = (x_0, y_0)$, $q_1 = (x_1, y_1) \in N \cap X^{\leqslant 0}$. It suffices to construct a continuous path τ: $[0, 1] \to N \cap X^{\leqslant 0}$ from q_0 to q_1 so that $\tau(t) \in N \cap X^{<0}$ whenever $0 < t < 1$. For $x \in \mathbf{R}^2$, let $h(x) = dx/\|x\|$ if $x \neq 0$, $h(0) = (d, 0)$. Notice that $f(h(x), y) < 0$ for all $(x, y) \in N$. The path τ can be chosen as the composition of the following four simple curves: First, move (x_0, y_0) linearly to $(h(x_0), y_0)$, then $(h(x_0), y_0)$ to $(h(x_1), y_0)$ through a rotation on the circle $\|x\| = d$ keeping y_0 fixed (here we are using index $\geqslant 2$), then $(h(x_1), y_0)$ linearly into $(h(x_1), y_1)$, and finally $(h(x_1), y_1)$ linearly to (x_1, y_1).

Lemmas 1 and 2 have the following consequence.

Lemma 3. *Assume the hypothesis of Lemma 1, and furthermore that any smoothly closed geodesic $c \in Q_0 \cap \Omega^{\leqslant 4r^2}$ (necessarily of length $2r$) has index $\geqslant 2$. Then $\overline{Q}_0 \cap \Omega^{\leqslant 4r^2}$ is the closure of $Q_0 \cap \Omega^{<4r^2}$, and a connected component of $\Omega^{\leqslant 4r^2}$.*

Proof. Let $p \in \bar{Q}_0 \cap \Omega^{\leqslant 4r^2}$, and let N be a neighborhood of p in $X = \Omega^{<b}$ for some $b > 4r^2 = a$ as in Lemma 2. We have $N \cap Q_0 \cap \Omega^{\leqslant 4r^2} \neq \varnothing$. But $N \cap \Omega^{<4r^2}$ is dense in $N \cap \Omega^{\leqslant 4r^2}$ and Q_0 open, so that also $N \cap Q_0 \cap \Omega^{<4r^2} \neq \varnothing$. Now $N \cap \Omega^{<4r^2}$ is connected, and by the lifting lemma, $Q \cap \Omega^{<4r^2}$ is a connected component of $\Omega^{<4r^2}$. Therefore $N \cap \Omega^{<4r^2} \subset Q_0$, and thus $N \cap \Omega^{\leqslant 4r^2}$ is contained in the closure of $Q_0 \cap \Omega^{<4r^2}$. This implies that $\bar{Q}_0 \cap \Omega^{\leqslant 4r^2}$ is the closure of the (connected) set $Q_0 \cap \Omega^{<4r^2}$, and is relatively open (and closed) in $\Omega^{\leqslant 4r^2}$, which completes the argument.

The following fact is basically standard.

Lemma 4 (*Connectedness lemma*). *Let f be a smooth proper function on a finite dimensional manifold X. Suppose, for some regular value b, all critical points of f in $X^{<b} - X^{\leqslant a}$ have index $\geqslant 2$ (but are possibly degenerate). Let $C_1, \dots, C_N$ be the connected components of $X^{\leqslant b}$. Then $C_1 \cap X^{\leqslant a}, \dots, C_N \cap X^{\leqslant a}$ are the connected components of $X^{\leqslant a}$. In particular, if $X^{\leqslant b}$ is connected, so is $X^{\leqslant a}$.*

Remark. (5) If in addition, all critical points in $f^{-1}(a)$ have index $\geqslant 2$, then $X^{\leqslant b}$ (path) connected implies that $X^{\leqslant a}$ is also path connected, since by Lemma 2, $X^{\leqslant a}$ is locally path connected.

Proof. Choose a decreasing sequence $b > a_k > a$, $\lim_{k\to\infty} a_k = a$, of regular values of f. By standard Morse theory, we can approximate f on $X^{\leqslant b}$ by a nondegenerate Morse function f_k which agrees with f on $X^{\leqslant a_k} \cup f^{-1}(b)$. If f_k is sufficiently close to f, all critical points of f_k in $X^{\leqslant b} - X^{\leqslant a_k}$ will have index $\geqslant 2$. Then it follows from the Morse inequalities that $H_i(X^{\leqslant b}, X^{\leqslant a_k}) = 0$ for $i = 0, 1$. Thus $C_1 \cap X^{\leqslant a_k}, \cdots, C_N \cap X^{\leqslant a_k}$ are the path connected components of $X^{\leqslant a_k}$. Since the intersection of a decreasing sequence of compact connected sets is connected, the lemma is proved.

Finally, we need

Lemma 5 (*Index lemma*). *Let M be odd dimensional with sectional curvature $0 < K \leqslant 1$. Then any nonconstant smoothly closed geodesic $c \in \bar{Q}_0 \cap \Omega^{\leqslant 4\pi^2}$ (necessarily of length 2π) has*[1] *index $\geqslant 2$.*

Proof. Recall that by standard index form comparison, $K \leqslant 1$ implies that for any geodesic in M of length π, at most the end points can be conjugate, and if this is the case, any Jacobi field vanishing at the end points looks like a Jacobi field on the Euclidean sphere of curvature 1, i.e., is a multiple of a parallel field. Therefore, in our situation, the hypothesis of Lemma 1 is satisfied for $r = \pi$, and the geodesic c has precisely two conjugate points for $t = 1/2$ and $t = 1$, at length π and 2π respectively.

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[1] Some further information has been communicated to us by T. Sakai.

We can find broken Jacobi fields $J_{\pm}$ along c, not identically zero, where J_+, J_- are Jacobi on $[0, 1/2]$, $[1/2, 1]$ respectively, and vanish otherwise. Now J_+ and J_- span a 2-dimensional space V on which the index form I is zero. If V does not intersect the null space N of I nontrivially, then the orthogonal projection of V on the negative eigenspace of I must be an injection which implies index $(c) \geqslant 2$. Otherwise, there is $0 \neq J \in V \cap N$. We are working in the free loop space, so N consists of periodic Jacobi fields. By the above comparison argument, we have $J(t) = \sin 2\pi t \cdot E(t)$, where E is a closed parallel field along c. Using the Synge argument, since M is odd dimensional and $K > 0$, M is orientable, and we conclude that there exists a second closed parallel field F normal to c. Then E and F span a 2-dimensional space on which I is negative definite.

We can now derive our main result.

Theorem 6. *Let M be a simply connected compact riemannian manifold of odd dimension n. Suppose the sectional curvature satisfies $1/4 \leqslant K \leqslant 1$. Then*

(a) $\Omega^{\leqslant 4\pi^2}$ *is the closure of* $\Omega^{<4\pi^2}$, *and is connected*;

(b) $\Omega^{<4\pi^2}$ *is contained in* Q_0, *and is (pathwise) connected*;

(c) $c \in \Omega^{\leqslant 4\pi^2}$ *implies either* $c \in Q_0$, *or both* $c_{1/2}$ *and* $c_{1/2}^-$ *are geodesics of length* π *with conjugate end points*;

(d) *any nonconstant smoothly closed geodesic in M has length $\geqslant 2\pi$ and index $\geqslant 2$.*

Proof. As described in the proof of Lemma 5, $K \leqslant 1$ implies that the hypothesis of Lemma 1 is satisfied for $r = \pi$. By the same comparison technique, since $1/4 \leqslant K$, any geodesic in M of length $> 2\pi$ has index $n - 1 \geqslant 2$.

We argue first that for any $a > 4\pi^2$, $\Omega^{\leqslant a}$ is connected. If not, let c_1, c_2 be curves in different connected components of $\Omega^{\leqslant a}$. Since M is simply connected, c_1 and c_2 are homotopic. After suitable refinement of the break point subdivision of the parameter interval $[0, 1]$, we can therefore join c_1 and c_2 by a continuous path in $\Omega^{\leqslant b}$ for some $b \geqslant a$. So c_1 and c_2 belong to the same connected component of $\Omega^{\leqslant b}$ for some regular value $b \geqslant a$, contradicting Lemma 4.

Using Lemma 4 again, we conclude that $\Omega^{\leqslant 4\pi^2}$ is connected. Now we apply Lemmas 5 and 3 to obtain that $\overline{Q}_0 \cap \Omega^{\leqslant 4\pi^2} = \Omega^{\leqslant 4\pi^2}$ is the closure of $Q_0 \cap \Omega^{<4\pi^2}$, which completes the proof of (a). In particular, $Q_0 \cap \Omega^{<4\pi^2}$ is dense, and by Lemma 1, connected and closed in $\Omega^{<4\pi^2}$. Thus $Q_0 \cap \Omega^{<4\pi^2} = \Omega^{<4\pi^2}$ is connected, which proves (b). The last two statements (c) and (d) are an immediate consequence of (a), (b), and Lemmas 1 and 5.

We conclude our discussion with some remarks. The result in (d) that all nonconstant smoothly closed geodesics have index $\geqslant 2$ is rather surprising. One has as immediate consequence (which is not the strongest conclusion that can be drawn) that for any $a \geqslant 0$, the loop spaces $\Omega^{<a}$ (and $\Omega^{<a}$ as well) are pathwise connected; cf. also Remark 5. The last statement holds also in even dimensions under the much weaker assumption $0 < K \leqslant 1$, since by the Synge Lemma, every nonconstant smoothly closed geodesic has index $\geqslant 1$. Clearly, Theorem 6 holds in that case except that in (d), the index is only $\geqslant 1$. Using the preceding remark, just Lemma 1 is needed for the proof. Finally, both in odd and even dimensions, Theorem 6(b) provides an obvious direct argument for the fact that the injectivity radius of the exponential map is $\geqslant \pi$.

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STATE UNIVERSITY OF NEW YORK, STONY BROOK